Reproducing Kernels and a Lower Bound for the Composition Operator on the Classical Hardy Space

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Abstract

Methods for computing the norm of the complex composition operator $C_\varphi$ on $H^2$ are unknown for general analytic $\varphi : \mathbb{D} \to \mathbb{D}$. However, the value $||C_\varphi||$ is known in some cases, and in other cases, lower bounds have been calculated. One such example is a particular rotational family of linear fractionals $\varphi_\theta$, for which T. Jones computed a lower bound for $C_{\varphi_\theta}$. We mimic the Jones computation with a new family of linear fractionals and attempt to find a lower bound analogous to the Jones lower bound.
1 Introduction

This section introduces the Hardy space, the reproducing kernels, and the composition operator on
the Hardy space. Once this is in place, we discuss lower bounds on the norm of the composition
operator, the linear fractional transformations which will be the focus of this paper, and the task at
hand in general.

1.1 Features of the Hardy space

Denote the open unit disk in the complex plane by $D = \{ z \in \mathbb{C} : |z| < 1 \}$. Define the Hardy
space, $H^2$, to be the set of all functions mapping from $D$ to $\mathbb{C}$ such that each function has a power
series representation with complex coefficients whose magnitudes are square summable. That is,
$f : D \to \mathbb{C}$ is in $H^2$ if there exists $\{ a_n \}_{n=0}^{\infty} \subseteq \mathbb{C}$ such that
\[
 f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
and
\[
 \sum_{n=0}^{\infty} |a_n|^2
\]
is convergent. One can check that $H^2$ is a vector space. We impose an inner product on this set,
declared by
\[
 \langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n b_n .
\]

Let $\| \cdot \|$ be the norm induced by the inner product, so that for $f \in H^2$, $\|f\| = (f, f)^{1/2}$. Equipped
with this inner product and the associated norm, $H^2$ becomes a Hilbert space of functions. Moreover,
the existence of power series representations for the functions in $H^2$ is equivalent to analyticity, so
each member of $H^2$ is an analytic function.

For a fixed $w \in D$, define the reproducing kernel at $w$, $K_w : D \to \mathbb{C}$, by
\[
 K_w(z) = \frac{1}{1 - \overline{w}z} .
\]
We wish to show that $K_w$ is a member of $H^2$. Note that $\frac{1}{1 - \overline{w}z}$ is the form of a geometric series
whose first term is 1, with geometric ratio $\overline{w}z$, i.e. $K_w(z) = \sum_{n=0}^{\infty} \overline{w}^n z^n$. Then we need only show
that the sum of the squared coefficients, $\sum_{n=0}^{\infty} (|\overline{w}|^n)^2$, converges. Observe that
\[
 \sum_{n=0}^{\infty} (|\overline{w}|^n)^2 = \sum_{n=0}^{\infty} |w|^{2n} = \frac{1}{1 - |w|^2} \in \mathbb{R} .
\]

Then we conclude that for each $w \in D$, the reproducing kernel $K_w$ lives in $H^2$, and further,
\[
 \|K_w\| = \frac{1}{\sqrt{1 - \|w\|^2}} .
\]
The reproducing kernels have a number of desirable qualities. The first of these is the “repro-
ducing” property: taking the inner product of the reproducing kernel at $w$ with any function $f$ from
$H^2$ gives $f(w)$. To see this, let $f \in H^2$. Then
\[ \langle f, K_w \rangle = (\sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} w^n z^n) = \sum_{n=0}^{\infty} a_n w^n = f(w). \]

Also notable is the fact that the reproducing kernels are dense in \( H^2 \). By this, we mean that the span of the reproducing kernels forms a subspace of \( H^2 \) whose orthogonal complement is trivial: supposing \( f \in (\text{span}\{K_w; w \in \mathbb{D}\})^\perp \), then \( f(w) = \langle f, K_w \rangle = 0 \) \( \forall w \in \mathbb{D} \iff f \) is the zero function. This property will be of great importance with respect to the next feature of \( H^2 \).

We now define the composition operator \( C_\varphi \). Let \( \varphi \) be a self-map of \( \mathbb{D} \). Then the composition operator is defined by \( C_\varphi(f) = f \circ \varphi \), where \( f \) is a function from \( H^2 \). We say that \( C_\varphi \) is bounded if there exists \( r \in \mathbb{R} \) such that \( \frac{||C_\varphi(f)||}{||f||} < r \) for all \( f \in H^2 \). In the case that \( C_\varphi \) is bounded, we can define the operator norm of the composition operator by

\[ ||C_\varphi||_{\text{op}} = \sup_{f \in H^2; ||f|| \neq 0} \frac{||C_\varphi(f)||}{||f||}. \]

\( C_\varphi \) is a linear function regardless of the choice of \( \varphi \), but we wish to restrict our choices of \( \varphi \) so that \( C_\varphi \) maps to \( H^2 \) and is thus a linear operator on the Hardy space. We appeal to a result by J. E. Littlewood [1] regarding this issue:

**Theorem.** If \( \varphi \) is an analytic self-map of \( \mathbb{D} \), then \( C_\varphi \) is a bounded linear operator on \( H^2 \).

In light of this theorem, it becomes clear that requiring that \( \varphi \) be an analytic self-map of \( \mathbb{D} \) forces the image of \( H^2 \) under \( C_\varphi \) to land within \( H^2 \). Then for analytic \( \varphi : \mathbb{D} \to \mathbb{D} \), \( C_\varphi \) is a linear operator on \( H^2 \). Furthermore, Littlewood’s theorem is useful in confirming that \( ||C_\varphi(f)|| \) is bounded, so the supremum exists and \( ||C_\varphi||_{\text{op}} \) is well-defined. Note that \( \varphi \) is analytic, and the sum of its squared coefficients is bounded, so the set of valid \( \varphi \) forms a subset of \( H^2 \).

With this structure in place, the problem then becomes computing the norm of \( C_\varphi \). This can be a difficult undertaking depending on one’s choice of \( \varphi \), and methods for computing \( ||C_\varphi|| \) for general \( \varphi \) are still not known. However, some facts have been discovered about \( ||C_\varphi|| \). The norm has been found for a select few types of \( \varphi \), and lower bounds have been found for other choices of \( \varphi \). One property of interest is the fact that the \( C_\varphi \) and its adjoint have the same norm. This property follows simply from the fact that \( C_\varphi \) is a linear operator on \( H^2 \), and we will make use of it to give a convenient lower bound in the next section.

Despite advances made on this problem, computing \( ||C_\varphi|| \) for general \( \varphi \) remains a daunting task. However, we can restrict our attention to certain classes of functions to somewhat simplify this problem. The computation to come involves a certain family of analytic self-maps of \( \mathbb{D} \), each of whose image is a disk internally tangent to the point 1 \( \in \mathbb{C} \). We further require that these maps fix the point 1. This family comes from a particular class of functions known as the linear fractionals, which are to be discussed in a later section.

### 1.2 Lower Bounds on \( ||C_\varphi|| \)

As was noted in the previous section, the fact that \( ||C_\varphi|| = ||C_\varphi^*|| \) allows us to use the reproducing kernels to find a lower bound for \( ||C_\varphi|| \). This lower bound is given by

\[ S^*_\varphi = \sup_{w \in \mathbb{D}} \frac{||C_\varphi^*(K_w)||}{||K_w||}. \]
As the reproducing kernels are a subset of $H^2$, it follows that $S^*_\varphi \leq ||C_\varphi||$. Further, knowing that the reproducing kernels are dense in $H^2$, we hope that they capture a lot of information about $H^2$ and give a bound which is somewhat close to the actual value of $||C_\varphi||$.

$S^*_\varphi$ is useful because the action of the adjoint of the composition operator on a reproducing kernel is simple to express algebraically. Manipulation of the inner product yields the identity $C^*_\varphi(K_w) = K_{\varphi(w)}$. Then

$$S^*_\varphi = \sup_{w \in \mathbb{D}} \frac{||K_{\varphi(w)}||}{||K_w||} = \sup_{w \in \mathbb{D}} \sqrt{\frac{1 - |w|^2}{1 - |\varphi(w)|^2}}.$$  

Depending on $\varphi$, this quantity may or may not be pleasant to compute. Perhaps a more intuitive approach would be to ignore the adjoint of $S$ and give a bound which is somewhat close to the actual value of $||C_\varphi||$. However, $S^*_\varphi$ is harder to work with than $S^*$. However, the rewards are greater: it turns out that $S^*_\varphi$ is another lower bound on $||C_\varphi||$.

**Proposition.** $S^*_\varphi \leq S^*_\varphi \leq ||C_\varphi||$.

**Proof.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Let $w \in \mathbb{D}$, and let $k_w$ be the normalized reproducing kernel at $w$, $k_w = \frac{K_w}{||K_w||}$. Then

$$||C^*_\varphi k_w|| = \sqrt{\frac{1 - |w|^2}{1 - |\varphi(w)|^2}}$$

$$= \sqrt{1 - |w|^2} \sqrt{1 - |\varphi(w)|^2} |K_{\varphi(w)}(\varphi(w))|$$

$$= \sqrt{1 - |w|^2} \sqrt{1 - |\varphi(w)|^2} \langle C_\varphi K_{\varphi(w)}, K_w \rangle$$

$$\leq \sqrt{1 - |w|^2} \sqrt{1 - |\varphi(w)|^2} ||C_\varphi K_{\varphi(w)}|| ||K_w||$$

$$= ||C_\varphi k_{\varphi(w)}||.$$  

It is therefore generally preferable to find $S^*_\varphi$ as it provides a more precise bound on $||C_\varphi||$.

### 1.3 Linear Fractional Transformations

Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. A linear fractional transformation $f$ is a function from $\mathbb{C}^*$ to $\mathbb{C}^*$ of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$, with $a$ and $b$ not both 0. Notice that all linear fractional maps are analytic except at the point $-\frac{d}{c}$. Given that we will attempt to compute $||C_\varphi||$ where $\varphi$ is a linear fractional, it is necessary only to consider those linear fractional maps with domain $\mathbb{D}$, and whose image is contained in $\mathbb{D}$. We will impose the restrictions mentioned at the end of the last section as well, but first discuss an important property that will make these stipulations more precise. It is an elementary property of linear fractional maps that circles and lines in the complex plane are carried to circles and lines. Then, given a linear fractional map $l$, the boundary of $\mathbb{D}$ is carried to a circle, and one
can show that $\mathbb{D}$ is mapped within that circle, so that the image of $\mathbb{D}$ under $l$ is an open disk. To reduce the scope of the overall problem to something manageable, we limit our attention to linear fractional self-maps $l$ of the disk such that $l(\mathbb{D}) \subseteq \mathbb{D}$, $l(\mathbb{D})$ lies tangent to $\mathbb{D}$ at the point 1, and $l(1) = 1$.

We now aim to find the explicit formula for such a map given the location of its image disk. Let $l$ be a linear fractional self-map of $\mathbb{D}$ which fixes 1 and which has image internally tangent to $\mathbb{D}$ at 1. Let $r$ denote the radius of the image ($r < 1$), let $c$ denote the center of the image (note that $c + r = 1$), and let $m$ be the distance between $c$ and $l(0)$. As it turns out, for fixed $\theta$ there is a unique linear fractional map with a particular $r$, $c$, and $m$. Then by fixing these three values and letting $\theta$ vary, we obtain a family of linear fractionals $\{l_\theta : \theta \in [0, 2\pi]\}$.

The particular family we will use is that whose image has radius $\frac{1}{3}$, center $\frac{2}{3}$, and $m$ value $\frac{1}{4}$. Let $\varphi_0(z)$ denote this family. By translating, dilating, and otherwise manipulating $\varphi_0(\mathbb{D})$, we can morph the image disk back to $\mathbb{D}$. Finding the maps that do this trick, then inverting and composing them gives a map that sends $\mathbb{D}$ to the desired image disk. This process is outlined in the T. Jones paper [2], so we will gloss over the details, but it is important to note that this method requires the use of an automorphism which we christen $\psi$, and this function differs from family to family. In the present example, $\psi(z) = \frac{4z - 3}{4 - 3z}$, whereas $\psi(z) = \frac{3z - 2}{3 - 2z}$ for the Jones family. This process eventually yields a “seed” map $\varphi_0(z) = \frac{10z + 11}{9z + 12}$, which in turn gives rise to the general $\varphi_\theta$,

$$
\varphi_\theta(z) = \frac{e^{i\theta}}{3} \left( \frac{4\psi(e^{-i\theta})z + 3}{4 + 3\psi(e^{-i\theta})z} \right) + \frac{2}{3}.
$$

By a result from Basor and Retsek [3], $\varphi_\theta$ can be expressed as

$$
\varphi_\theta(z) = \frac{(\varphi_\theta - 1)z + (\alpha_\theta + 1)}{\alpha_\theta z + \beta_\theta}.
$$

Some algebra yields

$$
\alpha_\theta = \frac{27 + 36\psi(e^{i\theta})}{14},
$$

$$
\beta_\theta = \frac{48 + 36\psi(e^{i\theta})}{14}.
$$

These values give

$$
\varphi_\theta(z) = \frac{(34 + 36\psi(e^{-i\theta}))z + (41 + 36\psi(e^{i\theta}))}{(27 + 36\psi(e^{-i\theta}))z + (48 + 36\psi(e^{i\theta}))}.
$$

The set $\{\varphi_\theta : \theta \in [0, 2\pi]\}$, abbreviated $\{\varphi_\theta\}$, is the rotational family we will be dealing with.

1.4 The Goal

Ideally, we would like to fully understand the composition operator $C_\varphi$. Such an understanding would begin with a general calculation of $||C_\varphi||$. Despite efforts to this end, however, a solution remains elusive: the quantity

$$
||C_\varphi|| = \sup_{f \in H^2} \frac{||f \circ \varphi||}{||f||}
$$

is too difficult to deal with in such a general form. To simplify, we can restrict our choices of $\varphi$ to obtain $C_\varphi$ for a subset of the analytic self-maps of $\mathbb{D}$, or we can restrict our choices of $f$ to obtain a lower bound on $||C_\varphi||$. Our computation of $S_{\varphi_\theta}$ will do both: we use the reproducing kernels instead.
of all of $H^2$, and apply this method specifically to the family $\{\varphi_\theta\}$ we have chosen. It may seem that $\{\varphi_\theta\}$ is a drop in the bucket compared to the collection of analytic $\varphi : \mathbb{D} \to \mathbb{D}$ for which $||C_\varphi||$ remains to be solved, but this investigation will hopefully prove more rewarding than it appears. Once a method for dealing with $\{\varphi_\theta\}$ has been established, it seems reasonable to expect that one could use it to find $S_{\varphi_\theta}$ for an arbitrary linear fractional map $\varphi_\theta$ which satisfies our conditions. It is with this goal in mind that we proceed with the calculation.

2 Computing $S_{\varphi_\theta}$

The maps $\varphi_\theta$ have been chosen, and an adequate lower bound has been found for $||C_\varphi||$. The task is now to compute $S_\varphi$. T. Jones has already calculated $S_\varphi$ with a different rotational family of linear fractional, so this process will mimic the process in his paper, and ideally will produce analogous results.

2.1 An Outline of the Process

The first step is to simplify the fraction in the definition of $S_\varphi$. This portion of the computation will deal with linear fractional transformations in general, so any mention of $\varphi$ or $\varphi_\theta$ refers to a general linear fractional map and not specifically to $\varphi_\theta$ from (1) above. Jones details the derivation of $S_\varphi$ in his paper, which ultimately concludes that

$$\frac{||C_{\varphi_\theta}(K_w)||}{||K_w||^2} = \frac{(|\alpha_\theta|^2 + 2\Re(\alpha_\theta \beta_\theta \gamma(w)) + |\beta_\theta|^2)(1 - |w|^2)}{|\beta_\theta - (\overline{\alpha_\theta} + 1)w|^2 - |\alpha_\theta - (\beta_\theta - 1)w|^2} \quad (2)$$

where

$$\gamma(w) = \frac{\alpha_\theta - w(\beta_\theta - 1)}{w(\overline{\alpha_\theta} + 1) - \overline{\beta_\theta}}.$$

Then to find $S_\varphi$, it is necessary to find a way to deal with this expression. More specifically, we must have a method to find the $w$ from the unit disk which maximizes the right hand side of (2), which we will refer to as $N(w)$. This expression is relatively tame except for the term $\Re(\alpha_\theta \beta_\theta \gamma(w))$, which will cause some difficulty. However, there is a convenient way of dealing with this term: we can examine its level curves, then consider $N(w)$ only over the points of $\mathbb{D}$ lying on a particular level curve, if there are any. A bit of analysis shows that the level curves of $\Re(\alpha_\theta \beta_\theta \gamma(w))$ are actually circles which partition the complex plane (in particular, they partition $\mathbb{D}$, and whose centers lie on a line.

Let $k \in \mathbb{R}$ and denote the set of $w$ in $\mathbb{D}$ such that $\Re(\alpha_\theta \beta_\theta \gamma(w)) = k$ by $\delta_k$. Call $k$ “relevant” if $\delta_k$ is nonempty. Geometrically, each $\delta_k$ for relevant $k$ is an arc from the circular level curve $\Re(\alpha_\theta \beta_\theta \gamma(w)) = k$ passing through the disk, and $\{\delta_k : k \text{ is relevant}\}$ partitions $\mathbb{D}$. Because of this, we can maximize $N(w)$ over $\delta_k$ for each relevant $k$, in each case treating the pesky term $\Re(\alpha_\theta \beta_\theta \gamma(w))$ as the constant $k$. This process will yield for each relevant $k$ a maximum value $M_k$. The only step remaining is to maximize $\{M_k\}$ over the relevant $k$, and we will have arrived at $S_\varphi$.

2.2 An Example: The Jones Computation

T. Jones carried this computation out with his own rotational family of linear fractionals, those induced by the values

$$\alpha_\theta = 4 + 6\psi(e^{i\theta})$$

$$\beta_\theta = 9 + 6\psi(e^{i\theta})$$
where

\[ \psi(z) = \frac{3z - 2}{3 - 2z}. \]

The formula for a general member of the rotational family induced by these \( \alpha_\theta \) and \( \beta_\theta \) is given by

\[ \varphi_\theta(z) = \frac{4e^{i\theta}}{5} \left( \frac{3\psi(e^{-i\theta})z + 2}{3 + 2\psi(e^{-i\theta})z} \right) + \frac{1}{5}. \]

Before proceeding any further, it is important to clarify some notation. To keep the expressions to follow from becoming too burdensome, \( s \) and \( c \) will be used to refer to \( \sin(\theta) \) and \( \cos(\theta) \), respectively. The variable \( c \) is not to be confused with the center of an image disk as it was used in Section 1.3. In fact, the center in that context is dependent on \( r \), and is mentioned primarily to aid in visualizing \( \varphi_\theta(D) \), so \( c \) will hereafter refer exclusively to \( \cos(\theta) \). Similarly, the variable \( s \) is from this point forward reserved for \( \sin(\theta) \). This will be the convention for the remainder of the discussion, including later sections.

The purpose of this section is to give some insight into how the steps in the process outlined above are going to work with a particular family of rotational maps. Thus, the discussion will be brisk, but the same discussion with our very own \( \{ \varphi_\theta \} \) will follow to fill in the details.

The Jones computation begins by declaring \( w = x + iy \), then finding \( \text{Re}(\alpha_\theta \beta_\theta \gamma(w)) \), which turns out to be

\[
\text{Re}(\alpha_\theta \beta_\theta \gamma(x + iy)) = \frac{-300}{13 - 12c} \left[ \frac{75 - 10(3 + 13c)x - 130sy + (51 + 26c)(x^2 + y^2)}{225 - 10(9 + 24c)x - 240sy + (73 + 48c)(x^2 + y^2)} \right].
\]

This value is called \( k \) to emphasize the fact that \( \text{Re}(\alpha_\theta \beta_\theta \gamma(w)) \) is to be treated as a constant, and we let

\[
\hat{k} = -\frac{(13 - 12c)k}{300} = \frac{75 - 10(3 + 13c)x - 130sy + (51 + 26c)(x^2 + y^2)}{225 - 10(9 + 24c)x - 240sy + (73 + 48c)(x^2 + y^2)}.
\]

Multiplying both sides by the denominator, rearranging terms, and completing the square affords the equivalent formulation

\[
\frac{625}{(51 + 26c - (73 + 48c)\hat{k})^2} = \left( x - \frac{(15 + 65c) - (45 + 120c)\hat{k}}{(51 + 26c) - (73 + 48c)\hat{k}} \right)^2 + \left( y - \frac{65s - 120s\hat{k}}{(51 + 26c) - (73 + 48c)\hat{k}} \right)^2,
\]

the familiar equation of a circle with radius

\[
R_\hat{k} = \frac{25}{|51 + 26c - (73 + 48c)\hat{k}|}
\]

and center

\[
C_\hat{k} = \left( \frac{(15 + 65c) - (45 + 120c)\hat{k}}{51 + 26c - (73 + 48c)\hat{k}}, \frac{65s - 120s\hat{k}}{51 + 26c - (73 + 48c)\hat{k}} \right).
\]

For a given \( \hat{k} \), the circle centered at \( C_\hat{k} \) with radius \( R_\hat{k} \) is exactly the set of values \( w \) for which \( \text{Re}(\alpha_\theta \beta_\theta \gamma(w)) = \hat{k} \). Next, the relevant \( \hat{k} \) values are found. Some analysis shows that the set of relevant \( \hat{k} \) values is
\[
\frac{13(1 + c)}{4(-19 + 6c)} < \hat{k} < \frac{-37 + 13c}{4(-19 + 6c)}.
\]

Re(\(\alpha_0\beta_0\gamma(w)\)) has been adequately dealt with, so we can return to the problem of maximizing \(N(w)\). For fixed \(\hat{k}\), the problem of maximizing \(N(w)\) over a given \(\delta_k\) is solvable using Lagrange multipliers. This method produces the result

\[
\sup_{w \in \delta_k} N(w) = \frac{(13 - 24\hat{k})(37 - 13c - (76 - 24c)\hat{k})}{100 - 200\hat{k}} \quad \text{when } \hat{k} > \frac{13c - 12}{-26 + 24c}
\]

and

\[
\sup_{w \in \delta_k} N(w) = \frac{(13 - 24\hat{k})(13 + 13c - (76 - 24c)\hat{k})}{100 + 200\hat{k}} \quad \text{when } \hat{k} < \frac{13c - 12}{-26 + 24c}.
\]

This leaves for each \(\hat{k}\) a single value, \(M_k\), which is the supremum of \(N(w)\) along \(\delta_k\). Taking the supremum of \(\{M_k : \hat{k} \text{ is relevant}\}\) gives the final result:

\[
S^2_{\varphi_0} = \begin{cases} 
\frac{25}{52 - 48 \cos(\theta)} & \text{if } 0 \leq \theta \leq \cos^{-1}\left(\frac{277}{278}\right) \\
\frac{25 - 12 \cos(\theta)}{2} - \sqrt{3}(1 - \cos(\theta)(19 - 6 \cos(\theta))) & \text{if } \cos^{-1}\left(\frac{277}{278}\right) \leq \theta \leq \pi
\end{cases}
\]

This concludes the computation of \(S_{\varphi}\) for the Jones family of linear fractional.

### 2.3 The \(\delta_k\) and Level Curves of \(\Re(\alpha_0\beta_0\gamma(w))\)

Now is finally the time to start pursuing \(S_{\varphi_0}\) for the \(\varphi_0\) from equation (1) at the end of Section 1.3. The computation to follow will proceed in the same fashion as the Jones computation, but we will include slightly more detail in each step for an understanding of the details as well as the process. First we obtain \(\gamma\) in the standard \(A + Bi\) form. Letting \(w = x + iy\) and viewing \(\gamma\) as a function of \(x\) and \(y\), we can find functions \(A\) and \(B\) such that we can write \(\gamma(x, y) = A(x, y) + B(x, y)\). Due to their length, the specific expressions of \(A\) and \(B\) have been omitted, but finding them is only a matter of rationalizing the denominator, then separating the real and imaginary parts. Similarly, \(\bar{\alpha_0\beta_0}\) can be written as \(C + Di\), so \(\Re(\alpha_0\beta_0\gamma(w)) = \Re((A + Bi)(C + Di)) = AC - BD\). This gives

\[
\Re(\bar{\alpha_0\beta_0}\gamma(w)) = \frac{27(-108 + 144x + 75cx - 60x^2 - 50cx^2 + 75sy - 60y^2 - 50cy^2)}{(25 - 24c)(144 - 192x - 72cx + 73x^2 + 48cx^2 - 72sy + 73y^2 + 48cy^2)}
\]

As promised, we set this expression equal to a constant \(k\), then examine which \(x\) and \(y\) make the equation hold true. Noting that the constants may be removed and added in later without affecting the values of \(x\) and \(y\), we work with the slightly simpler equation

\[
\hat{k} = \frac{(25 - 24c)k}{27} = \frac{(-108 + 144x + 75cx - 60x^2 - 50cx^2 + 75sy - 60y^2 - 50cy^2)}{(144 - 192x - 72cx + 73x^2 + 48cx^2 - 72sy + 73y^2 + 48cy^2)}.
\]

Multiplying both sides by the denominator, then separating constant terms and powers of \(x\) or \(y\) on either side of the equation, we get

\[
108 - 144\hat{k} = (144 + 75c - 192\hat{k} - 72c\hat{k})x + (-60 - 50c + 73\hat{k} + 48c\hat{k})x^2 + (75s - 72s\hat{k})y + (-60 - 50c + 73\hat{k} + 48c\hat{k})y^2.
\]

The \(x^2\) and \(y^2\) terms have the same coefficients, so divide both sides of the equation by \((-60 - 50c + (73 + 48c)\hat{k})\), then complete the square with respect to both \(x\) and \(y\) to get
\[
\left(\frac{21}{2(-60 - 50c + (73 + 48c)\hat{k})}\right)^2 = \left(x + \frac{144 + 75c - 192\hat{k} - 72\hat{k}}{2(-60 - 50c + (73 + 48c)\hat{k})}\right)^2 + \left(y + \frac{75s - 72s\hat{k}}{2(-60 - 50c + (73 + 48c)\hat{k})}\right)^2,
\]

once again matching the form of a circle with radius
\[ R_{\hat{k}} = \frac{21}{2|60 + 50c - (73 + 48c)\hat{k}|} \]
and center
\[ C_{\hat{k}} = \left(\frac{144 + 75c - 192\hat{k} - 72\hat{k}}{2(60 + 50c - (73 + 48c)\hat{k})}, \frac{75s - 72s\hat{k}}{2(60 + 50c - (73 + 48c)\hat{k})}\right). \]

This shows that the level curves of \( \text{Re}(\alpha\theta\beta\gamma(w)) \) are indeed circles. One can check that the centers of the circles lie on a line. Additionally, for any point \( z_0 \) in the complex plane, \( \text{Re}(\alpha\theta\beta\gamma(z_0)) \) takes on some value, and thus \( z_0 \) lies on one of these circles. Further, the circles are disjoint because it cannot be the case that \( \text{Re}(\alpha\theta\beta\gamma(z_0)) \) gives two different outputs for a single \( z_0 \). Then the circles form a partition of \( \mathbb{C} \), as claimed earlier, and the intersection of these circles with \( \mathbb{D} \) will indeed give a partition of \( \mathbb{D} \).

The next step is nailing down the relevant \( \hat{k} \) values. Let \( \text{Circ}(\hat{k}) = \{ z \in \mathbb{C} : \text{Re}(\alpha\theta\beta\gamma(z)) = \hat{k}\} \). In other words, \( \text{Circ}(\hat{k}) \) is just the circle having center \( C_{\hat{k}} \) and radius \( R_{\hat{k}} \). If \( C_{\hat{k}} \) lies outside of \( \mathbb{D} \), we see that \( \text{Circ}(\hat{k}) \) intersects \( \mathbb{D} \) if and only if
\[ |C_{\hat{k}}| - 1 < R_{\hat{k}} < |C_{\hat{k}}| + 1. \tag{4} \]

The left inequality ensures that \( \text{Circ}(\hat{k}) \) reaches far enough to contain at least part of the disk; the right ensures that \( \text{Circ}(\hat{k}) \) is not too large to miss \( \mathbb{D} \) altogether. Finding \( |C_{\hat{k}}| \) and solving these equations simultaneously gives the solution set
\[
\begin{align*}
-27 + 50c &< \hat{k} < -69 + 50c \quad \text{for} \quad -1 \leq c \leq -\frac{109}{144} \tag{5a} \\
-27 + 50c &< \hat{k} < \frac{60 + 50c}{73 + 48c} \quad \text{for} \quad -\frac{109}{144} < c \leq -\frac{37}{48} \tag{5b} \\
-27 + 50c &< \hat{k} < \frac{-69 + 50c}{-71 + 48c} \quad \text{for} \quad \frac{37}{48} < c \leq 1 \tag{5c}
\end{align*}
\]

This is the point at which our example diverges from the Jones example. In our computation, the expressions for the right endpoints of the intervals of \( \hat{k} \) change depending on where \( c \) is. This asserts more than a dependence on \( c \); this says that the function that governs the right endpoint for a given \( c \) value may be distinct from the function that governs the right endpoint for a different \( c \). Keeping in mind that \( c = \cos(\theta) \), it seems that for our \( \varphi_\theta \), the \( \hat{k} \) ranges are governed by different functions depending on the subinterval of \([0, 2\pi]\) in which \( \theta \) lies. If we wish to continue our investigation with no loss of generality, i.e. pursuing \( S_{\varphi_\theta} \) for all \( \theta \), it will be necessary to analyze the problem in cases.
2.4 \(N(w)\) and Lagrange Multipliers

Despite the setback mentioned above, the relevant \(\hat{k}\) values are now in hand and we can contend with \(N(w)\). Substituting \(\alpha\theta\) and \(\beta\theta\) as necessary, as well as \(Re(\alpha\theta;\gamma(w))\), we find that for this example,

\[
N(w) = \frac{3(144 - 192x + 55x^2 + 55y^2)}{(3 - 4x + x^2 + y^2)(144 - 192x + 72cx - 72sy + (73 + 48c)(x^2 + y^2))}.
\]

Note that in each term, the coefficient of \(x^2\) coincides with that of \(y^2\), so we can simplify the expression of \(N(w)\) by finding an expression for \(x^2 + y^2\). Solving for this value from the circle equation above gives

\[
N(w) = 3(25 - 24\hat{k})(60 + 50c - (73 + 48c)\hat{k})
\]

Define

\[
f(x, y) = \frac{60 + 50c - (73 + 48c)\hat{k}}{72 - 96x + c(25 - 24\hat{k})(6 - 5x) + 75sy + \hat{k}(-75 + 100x - 72sy)}
\]

and

\[
g(x, y) = \left( x + \frac{144 + 75c - 192\hat{k} - 72\hat{k}}{2(-60 - 50c + (73 + 48c)\hat{k})} \right)^2 + \left( y + \frac{75s - 72s\hat{k}}{2(-60 - 50c + (73 + 48c)\hat{k})} \right)^2.
\]

Note that \(f\) is just our newer formulation of \(N(w)\) with the constant factors removed, and \(g\) is the right side of the circle equation. We wish to use Lagrange multipliers to maximize \(f(x, y)\) subject to \(g(x, y) = \frac{-21}{2(60+50c-(73+48c)k)}\). We find that

\[
\frac{\partial f}{\partial x} = \frac{-96 + 100\hat{k} + 5c(25 - 24\hat{k})(60 + 50c - (73 + 48c))}{(72 - 96x + c(25 - 24\hat{k})(6 - 5x) + 75sy + \hat{k}(-75 + 100x - 72sy))^2}
\]

\[
\frac{\partial f}{\partial y} = \frac{-75s + 72s\hat{k}(60 + 50c - (73 + 48c))}{(72 - 96x + c(25 - 24\hat{k})(6 - 5x) + 75sy + \hat{k}(-75 + 100x - 72sy))^2}
\]

and

\[
\frac{\partial g}{\partial x} = 2\left( x + \frac{144 + 75c - 192\hat{k} - 72\hat{k}}{2(-60 - 50c + 73k + 48ck)} \right)
\]

\[
\frac{\partial g}{\partial y} = 2\left( y + \frac{75s - 72s\hat{k}}{2(-60 - 50c + 73k + 48ck)} \right).
\]

We now want to find \(\lambda\) such that

\[
\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x},
\]

\[
\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y},
\]

and
At this point, some reasoning pertaining to the terms that comprise the partial derivatives of \( f \) and \( g \) must be used to show that certain qualities hold for such a \( \lambda \). For example, the Jones version of this step notes that neither \( \frac{\partial f}{\partial x} \) nor \( \frac{\partial g}{\partial x} \) vanishes, so \( \lambda \) must be nonzero and the argument proceeds from there. It is believable that similar conclusions can be drawn regarding our system of equations, but the fact that the expression for the interval of relevant \( \hat{k} \) changes based on the value of \( c \) complicates this analysis considerably. Thus, we shall defer the full breakdown of the problem to a later time, and instead discuss what has happened so far and what is still to come.

3 The Road Ahead

We have established the framework necessary for carrying out the Lagrange multiplier method to maximize \( N(w) \). While this problem is no obstacle in theory, especially considering the fact that we have the Jones computation as a template for how this step should go, the fact is that the separation of the interval \([-1, 1]\) into the three subintervals discussed above will prove to be a large practical impediment. Nevertheless, the computation should proceed in a somewhat predictable manner. With the exception of the \( c \)-dependent expressions for intervals of relevant \( \hat{k} \), the direction of our computation has followed that of the Jones computation exactly. This seems to be no coincidence; the details included in this paper do not show it, but multiple algebraic miracles have contributed to the relative simplicity of various expressions that we have used to carry out the computation. In particular, the final forms of \( \alpha_\theta \) and \( \beta_\theta \) are especially trim. Some basic algebra within the context of a general rotational family \( l_\theta \) shows that

\[
\alpha_\theta = \frac{m^2 + mr\psi(e^{i\theta})}{(r - 1)(m^2 - r^2)}
\]

and

\[
\beta_\theta = \frac{r^2 + mr\psi(e^{i\theta})}{(r - 1)(m^2 - r^2)},
\]

where

\[
\psi(z) = \frac{rz - m}{r - mz}
\]

is once again the automorphism required to ensure that \( l(0) \) is a distance of \( m \) from the center of \( l(\mathbb{D}) \). Also convenient is the fact that the level curves of \( \text{Re}(\alpha_\theta\beta_\gamma(w)) \) ended up being circles whose centers lie on a line and whose radii vary continuously as a function of \( \hat{k} \). We predict that this will hold true for any linear fractional satisfying the requirements set forth in Section 1.3. If this is the case, we can employ the method used by Jones for the general calculation.

The Lagrange multiplier process will give for each \( \hat{k} \) a number of critical points of \( N(w) \), each of which lies on \( \text{Circ}(\hat{k}) \). We then determine which of these lie inside \( \mathbb{D} \), if any, and determine which from this collection give the largest value of \( N(w) \). By doing this, we get a single maximum value of \( N(w) \) for each \( \hat{k} \), and taking the supremum of all of these as \( \hat{k} \) ranges across its relevant values, we arrive at \( S_{\varphi_\theta} \) surprisingly quickly. The Jones computation concluded that each \( \text{Circ}(\hat{k}) \) contained two critical points, one lying inside \( \mathbb{D} \) and one lying outside. Conveniently, the point lying inside turned out to be a maximum, so no other analysis of points on \( \delta_\hat{k} \) was required. It seems reasonable to expect the same result for our rotational family; after all, the Jones \( \{\varphi_\theta\} \) and our \( \{\varphi_\theta\} \) are really not all that different. However, as we have seen, the objects in our computation may not act
exactly as they did for Jones, so we must be careful when carrying this part out. It could be that all critical points lie outside the unit disk. There could be more than two critical points. We may even have different combinations of these possibilities for different $\hat{k}$, requiring us to break each case into subcases! We see that our investigation has come close to attaining $S_{\phi_\theta}$ in theory, but actually grinding out the details will require some effort and careful attention to detail.

As a final note, we address the $\hat{k}$ values and their peculiar behavior as $c$ moves across $[-1, 1]$. It is relieving to find that (5a) and (5b) agree at their meeting point $-\frac{109}{144}$, as do (5b) and (5c) at their meeting point $\frac{37}{48}$. This means that the $\hat{k}$ values are given by a continuous function of $c$, but it is unclear why the functions governing the $\hat{k}$ range would change within the interval where there was no such anomaly in the Jones computation. To answer this question, we examine the equation (4), from which the $\hat{k}$ values (5a), (5b), and (5c) came. (4) is an inequality in the variables $c$ and $\hat{k}$, and because $c = \cos(\theta)$, we imposed the constraint $-1 \leq c \leq 1$. When we remove this restriction and treat (5a), (5b), and (5c) as functions of $c$, where $c$ can take any value in $\mathbb{R}$, we find that the solutions occur in “chunks” as they did with $c \in [-1, 1]$. The solutions of $\hat{k}$ break $\mathbb{R}$ into components, each of which has an associated $\hat{k}$ range whose endpoints are functions of $c$, and such that for any component, at least one of the expressions governing the right and left endpoints differs from that of any of the adjacent components. In other words, when we pay attention to solutions to (4) on all of $\mathbb{R}$, we see that our original results persist both inside this interval and outside. Freeing the Jones version of (4) from the constraint $c \in [-1, 1]$ shows that the solutions again break $\mathbb{R}$ into components. However, the interval of interest, $[-1, 1]$, lies entirely within a component, so it would initially appear as though the functions governing the $\hat{k}$ intervals do not depend on $\theta$.

The mechanism which governs where the breaks in components occurs is not known, so we do not know whether their movements are continuous, whether the number of components can change, etc. This is bittersweet news for us. On one hand, it shows that the results of solving (4) for the Jones example and our example are not actually as different as they first appeared. On the other hand, it confirms that for some rotational families, the calculation will be more labor intensive than the Jones computation. Considering an arbitrary rotational linear fractional family, it may or may not be the case that $[-1, 1]$ lies entirely within a component. Presently, we cannot rule out the possibility that for some rotational family, the interval $[-1, 1]$ contains a large, possibly infinite, number of components, each of which would require individual consideration to complete the computation. The plan from this point is to push through the Lagrange multiplier step of our computation to find $S_{\phi_\theta}$, then attempt to answer some of these questions to determine the best way to approach $S_{\phi_\theta}$ in the general case.
References

